Assignment 1 – Solution sheets

EC1 146

Question 1: Four uses of water:

1) Thermal power stations use water to cool down the internal systems of power generation and thus to avoid overheating. They take water from a neighboring body and then they release it at a higher temperature. We can apply numerical techniques to study if there is risk that the out flow of the stations (heated water) reaches the intake, because this could lead to a catastrophe.

2) Intakes for potable water in rivers are designed to take the "best" layers of water; i.e., the volumes that are less polluted. We can optimize the design through numerical techniques such that this objective is accomplished.

3) Mercury in SF Bay: the bay is polluted by traces of mercury in the water and the sediments. We can use numerical techniques to assess where the mercury from the sediments go when there are dredging operations in the bay.

4) Density currents are underflows that can transport contaminants for very long distances.

We can use numerical techniques to determine exactly the interface between salty and fresh water in the SF Bay. Thus, we can infer conclusions about the transport of pollutants in the bay.

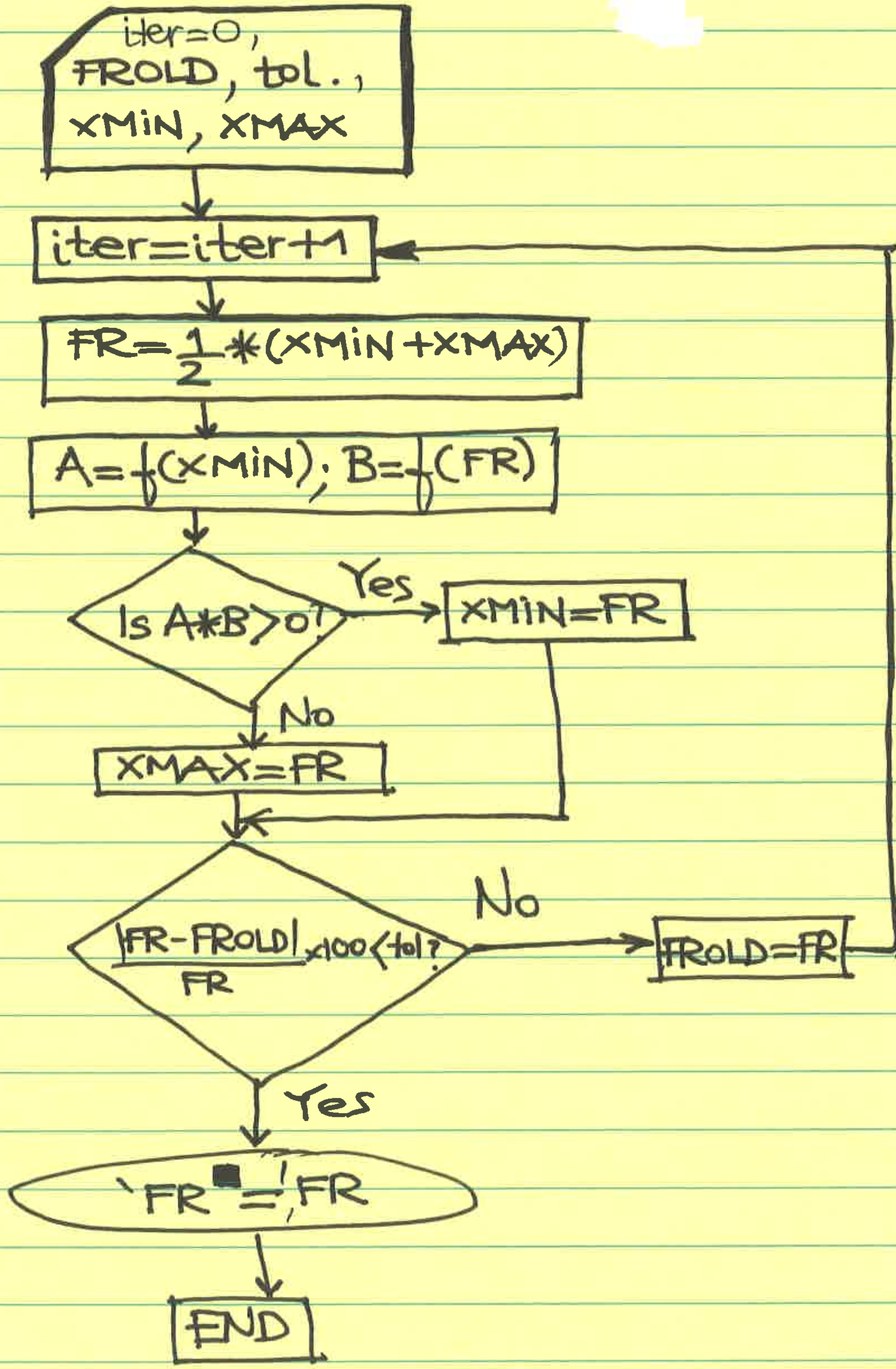
In 1): a) Problem: evaluate risk of collapse of power plants in 2D
b) Equation: advection/diffusion (heat eqn.)
c) Solution: finite differences with centered method for diffusion

In 2): a) Problem: evaluate best layers in a reservoir in 3D
b) Equation: advection/diffusion (heat eqn.)
c) Solution: finite differences with centered method for diffusion

In 3): a) Problem: evaluate mercury distribution in the SF Bay in 3D.
b) Equation: advection/diffusion (heat eqn.)
c) Solution: finite differences with centered method for diffusion.

In 4): a) Problem: evaluate density distribution in the SF Bay in 3D.
b) Equation: advection/diffusion
c) Solution: finite differences with centered method for diffusion.

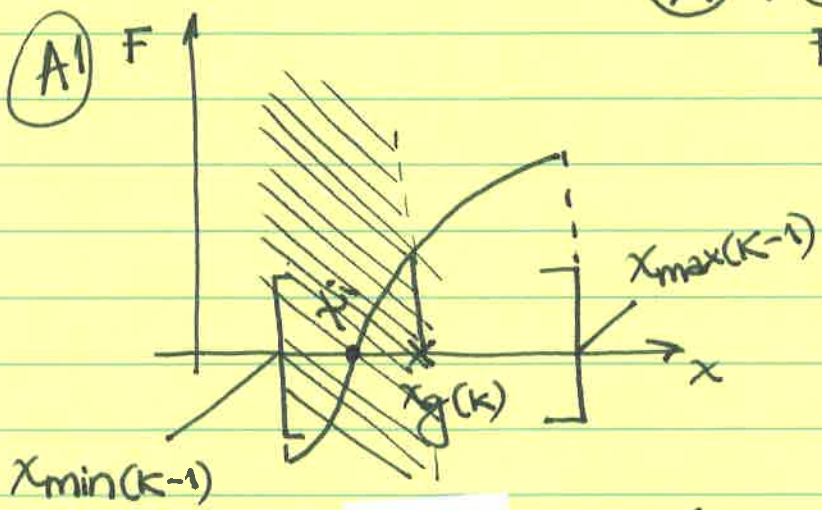
Question 2: Bisection of a general, implicit algebraic equation, $F(x_i)=0$



The rationale behind this is that the four cases we saw in class (A1, A2, B1 and B2), can be reduced to two if we consider the product between the values of the function evaluated at XMIN and FR. If the product is positive, then we need to update XMIN; if it is negative, we need to update XMAX.

(A) $F(x_{min(k-1)}) < 0$ & $F(x_{max(k-1)}) > 0$

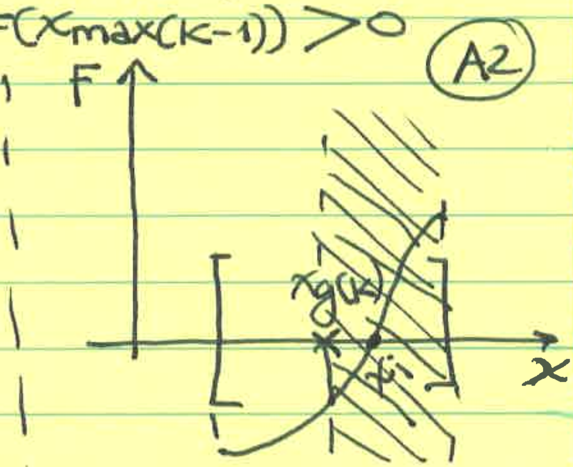
A1



$F(x_{min(k-1)}) * F(x_{g(k)}) < 0$

Next interval: $[x_{min(k-1)}, x_{g(k)}]$

A2



$F(x_{min(k-1)}) * F(x_{g(k)}) > 0$

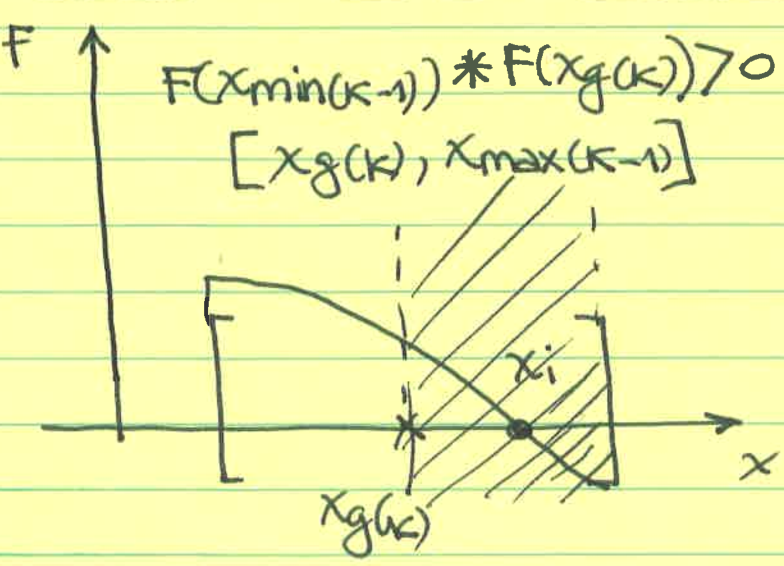
$[x_{g(k)}, x_{min(k-1)}]$

B1

$F(x_{min(k-1)}) > 0$ & $F(x_{max(k-1)}) < 0$

$F(x_{min(k-1)}) * F(x_{g(k)}) > 0$

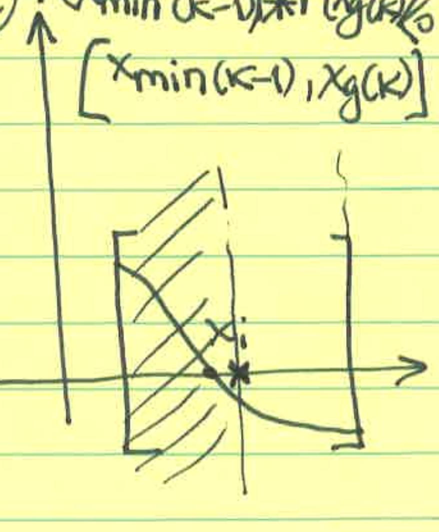
$[x_{g(k)}, x_{max(k-1)}]$



B2

$F(x_{min(k-1)}) * F(x_{g(k)}) < 0$

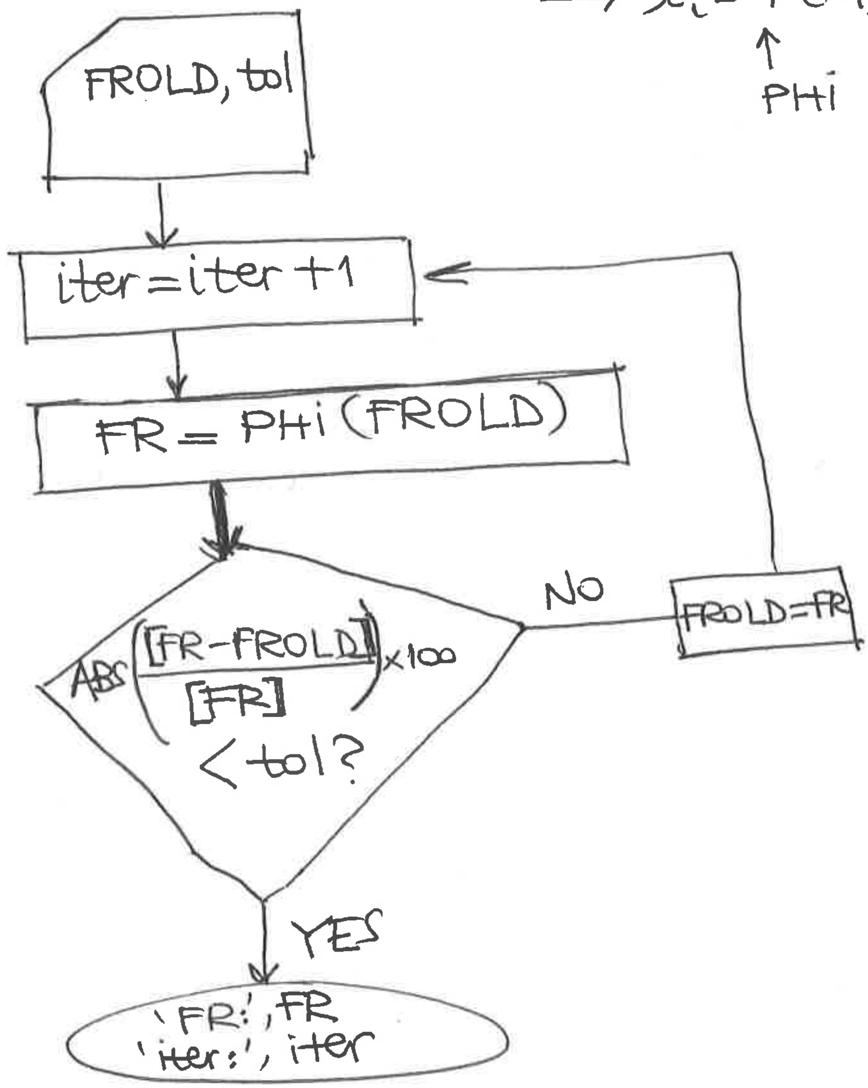
$[x_{min(k-1)}, x_{g(k)}]$



Fixed point for a general, implicit algebraic equation, $F(x_i)=0$

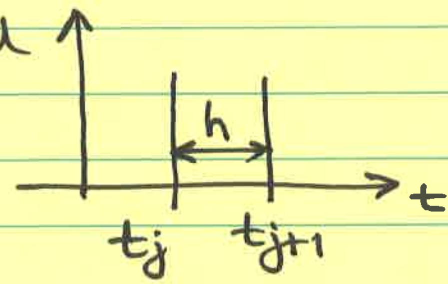
$$\Longrightarrow x_i = \varphi(x_i)$$

↑
PHI



Question 3

$$\frac{du}{dt} = -Ku \qquad K>0$$



$$\frac{u_{j+1}-u_j}{\Delta t} = \frac{u_{j+1}-u_j}{h} = -K u_{j+1} \quad \text{backward}$$

$$\Longrightarrow \quad u_{j+1} = u_j - Kh\, u_{j+1}$$

$$u_{j+1}(1+Kh) = u_j$$

$$u_{j+1} = \frac{u_j}{(1+Kh)}$$

Since K is positive and h is positive, u_{j+1} is always a decreasing function.

Then, it is stable all the time.

Accuracy will depend on the value of h. For $K=1$, $u_0=1$, h of the order of 0.05 gives accurate solutions.

Question 4

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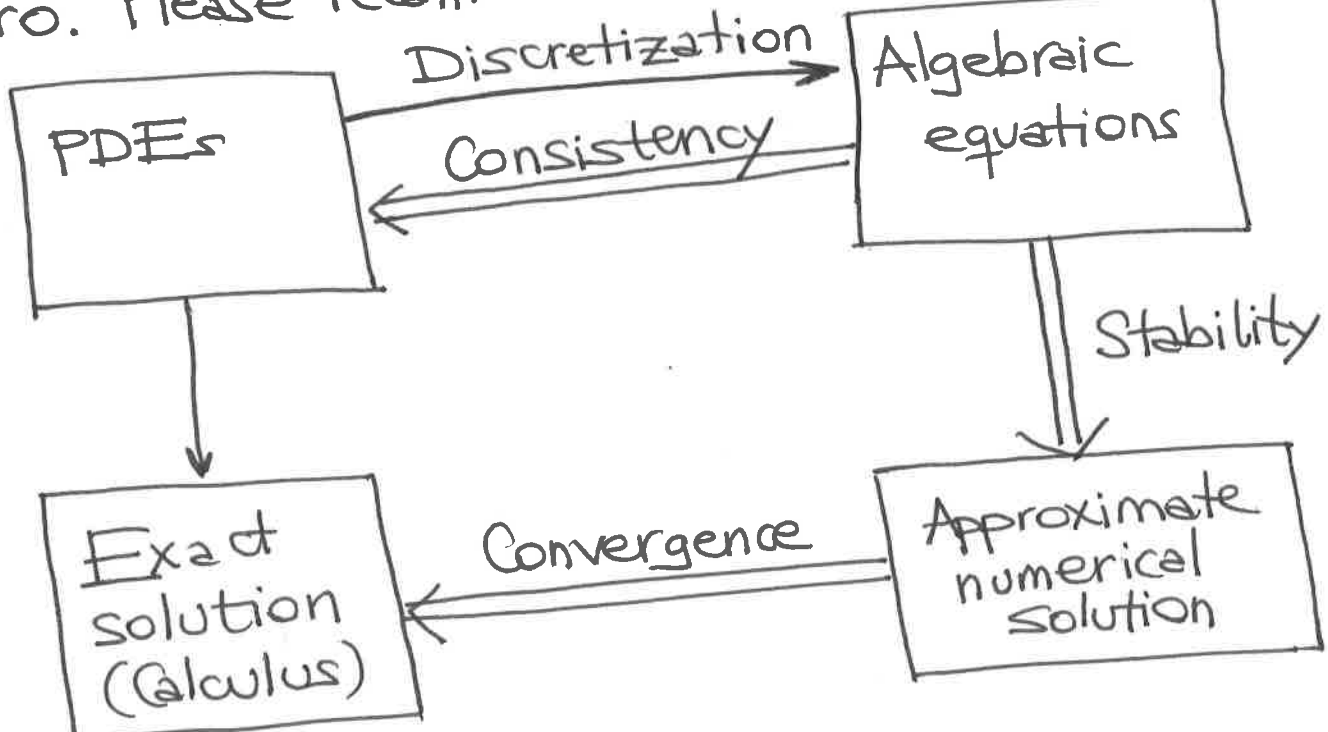
Order of convergence in the solution of an algebraic implicit equation indicates the "speed" at which the succession of values representing the iterative process approaches the actual solution. Mathematically:

$$\lim_{K \to \infty} \frac{|x_{g(K)} - x_i|}{|x_{g(K-1)} - x_i|^p} = \text{constant}$$

p indicates the rate of convergence, k is the iteration number, and x_i the root.

Convergence of ODEs: In ODEs, we say that a solution is convergent if it is "close" to the "true" solution when Δx and Δt approach zero. Please recall:



Question 5

Expanding in Taylor series at t_1

$$u(t_2) = u(t_1) + u'(t_1)\frac{(t_2-t_1)}{1!} + u''(t_1)\frac{(t_2-t_1)^2}{2!}$$

$$+ u'''(t_1)\frac{(t_2-t_1)^3}{3!} + \dots$$

Since $t_2 - t_1 = h$:

$$u(t_2) - u(t_1) = u'(t_1)\,h + u''(t_1)\frac{h^2}{2} + u'''(t_1)\frac{h^3}{3!}$$

$+ HOT$

Dividing by h:

$$\frac{u(t_2)-u(t_1)}{h} - u'(t_1) = \frac{u''(t_1)}{2}\,h + u'''(t_1)\frac{h^2}{6} + \dots$$

or

$$\left|\frac{u(t_2)-u(t_1)}{h} - u'(t_1)\right| \approx |u''(t_1)|\frac{h}{2}$$

Assuming that $|u''(t_1)| \leqslant C_1$ (bounded), we find

$$|\text{error}_{\text{Forward}}| \leqslant C_1 \frac{h}{2},$$ which is first order

Question 6

C-W equation:

$$\frac{1}{\sqrt{f}} + 0.869 \ln\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\sqrt{f}}\right) = 0$$

If $X = 1/\sqrt{f}$:

$$X = -0.869 \ln\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re} X\right) = \varphi(X)$$

$$f \in [1\times10^{-6}, 0.1]$$

$$X \in [1000, 3.16228]$$

$$|\varphi'(X)| = \frac{+0.869 \left(\frac{2.51}{Re}\right)}{\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re} X\right)}$$

Ex.: $Re = 3\times10^{6}$
$\varepsilon/D = 8\times10^{-4}$

$$\left.\begin{array}{l} |\varphi'(X)|_{1000} = 6.97\times10^{-4} \\ |\varphi'(X)|_{500} = 1.15\times10^{-3} \\ |\varphi'(X)|_{100} = 2.42\times10^{-3} \\ |\varphi'(X)|_{10} = 3.24\times10^{-3} \\ |\varphi'(X)|_{3.16228} = 3.32\times10^{-3} \end{array}\right\} \Longrightarrow m = 0.01$$

which is larger than all

$\Longrightarrow |\varphi'(X)| \ll m$, with $0 < m < 1 \Longrightarrow$ We know that the iterative process will converge from the theorem

Question 8

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F(X)=0

FUNC

DER=DERIVATIVE