HAND OUT 18: Hydraulic routing (Chapter 6 of our syllabus). Source: Mays, L. (2006). *"Water resources engineering."* John Wiley and Sons.

Check to see if $C_1 + C_2 + C_3 = 1$:

$0.3396 + 0.6038 + 0.0566 = 1$

Using equation (9.3.6) with $I_1 = 0$ cfs, $I_2 = 800$ cfs, and $Q_1 = 0$ cfs, compute Q_2 at $t = 1$ hr:

$$Q_2 = C_1I_2 + C_2I_1 + C_3Q_1$$
$$= (0.3396)(800) + 0.6038(0) + 0.0566(0)$$
$$= 272 \text{ cfs } (7.7 \text{ m}^3/\text{s})$$

Next compute Q_3 at $t = 2$ hr:

$$Q_3 = C_1I_3 + C_2I_2 + C_3Q_2$$
$$= (0.3396)(2000) + 0.6038(800) + 0.0566(272)$$
$$= 1178 \text{ cfs } (33 \text{ m}^3/\text{s})$$

The remaining computations result in

Time (hrs)	0	1	2	3	4	5	6	7
Q (cfs)	0	272	1178	2701	4455	4886	4020	3009

Time (hrs)	8	9	10	11	12	13	14	15
Q (cfs)	2359	1851	1350	918	610	276	16	1

9.4 HYDRAULIC (DISTRIBUTED) ROUTING

Distributed routing or *hydraulic routing*, also referred to as *unsteady flow routing*, is based up the one-dimensional unsteady flow equations referred to as the *Saint–Venant equations*. T hydrologic river routing and the hydrologic reservoir routing procedures presented previously a lumped procedures and compute flow rate as a function of time alone at a downstream locati Hydraulic (distributed) flow routings allow computation of the flow rate and water surface ele tion (or depth) as function of both space (location) and time. The Saint–Venant equations are p sented in Table 9.4.1 in both the *velocity-depth (nonconservation) form* and the *discharge-a (conservation) form.*

The momentum equation contains terms for the physical processes that govern the flow mome tum. These terms are: the *local acceleration term*, which describes the change in momentum d to the change in velocity over time, the *convective acceleration term*, which describes the chan in momentum due to change in velocity along the channel, the *pressure force term*, proportio to the change in the water depth along the channel, the gravity force term, proportional to the b slope S_0, and the friction force term, proportional to the friction slope S_f. The local and convecti acceleration terms represent the effect of inertial forces on the flow.

Alternative distributed flow routing models are produced by using the full continuity equati while eliminating some terms of the momentum equation (refer to Table 9.4.1). The simplest d tributed model is the *kinematic wave model*, which neglects the local acceleration, convecti acceleration, and pressure terms in the momentum equation; that is, it assumes that $S_0 = S_f$ and t friction and gravity forces balance each other. The *diffusion wave model* neglects the local a convective acceleration terms but incorporates the pressure term. The *dynamic wave model* co siders all the acceleration and pressure terms in the momentum equation.

The momentum equation can also be written in forms that take into account whether the fl is steady or unsteady, and uniform or nonuniform, as illustrated in Table 9.4.1. In the continu equation, $\partial A/\partial t = 0$ for a steady flow, and the lateral inflow q is zero for a uniform flow.

Table 9.4.1 Summary of the Saint–Venant Equations*

Continuity equation

Conservation form $$\frac{\partial Q}{\partial x}+\frac{\partial A}{\partial t}=0$$

Nonconservation form $$V\frac{\partial y}{\partial x}+\frac{\partial V}{\partial x}+\frac{\partial y}{\partial t}=0$$

Momentum equation

Conservation form

$$\frac{1}{A}\frac{\partial Q}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + g\frac{\partial y}{\partial x} - g(S_0 - S_f) = 0$$

Local acceleration term	Convective acceleration term	Pressure force term	Gravity force term	Friction force term

Nonconservation form (unit with element)

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - g(S_0 - S_f) = 0$$

Kinematic wave (gravity and friction force terms)
Diffusion wave (pressure, gravity and friction force terms)
Dynamic wave (all terms)

*Neglecting lateral inflow, wind shear, and eddy losses, and assuming $\beta = 1$.

x = longitudinal distance along the channel or river, t = time, A = cross-sectional area of flow, h = water surface elevation, S_f = friction slope, S_0 = channel bottom slope, g = acceleration due to gravity, V = velocity of flow, and y = depth of flow.

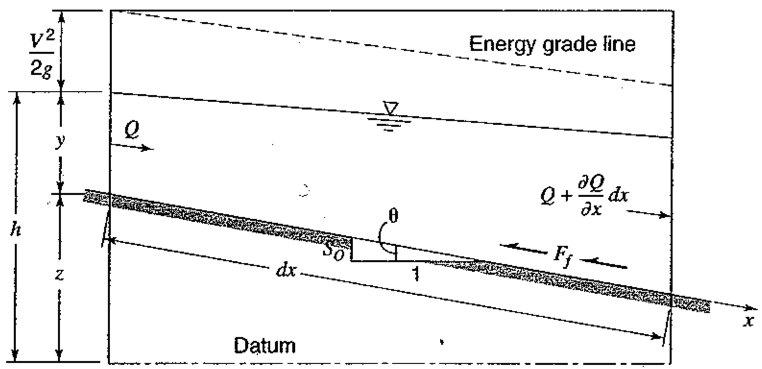
9.4.1 Unsteady Flow Equations: Continuity Equation

The *continuity equation* for an unsteady variable-density flow through a control volume can be written as in equation (3.2.1):

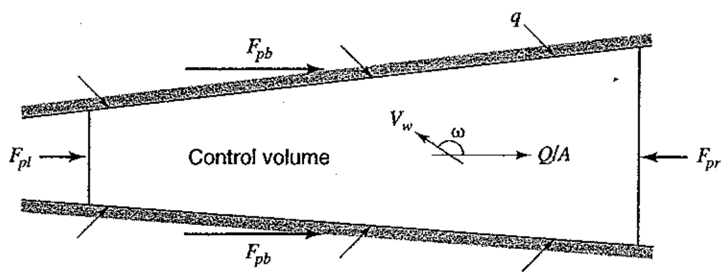
$$0=\frac{d}{dt}\int_{CV}\rho\, d\forall+\int_{CS}\rho\mathbf{V}\cdot d\mathbf{A} \tag{9.4.1}$$

Consider an elemental control volume of length dx in a channel. Figure 9.4.1 shows three views of the control volume: (*a*) an elevation view from the side, (*b*) a plan view from above, and (*c*) a channel cross-section. The inflow to the control volume is the sum of the flow Q entering the control volume at the upstream end of the channel and the lateral inflow q entering the control volume as a distributed flow along the side of the channel. The dimensions of q are those of flow per unit length of channel, so the rate of lateral inflow is qdx and the mass inflow rate is

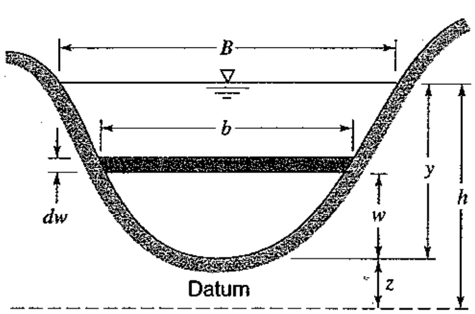
$$\int_{\text{inlet}}\rho\mathbf{V}\cdot d\mathbf{A}=-\rho(Q+qdx) \tag{9.4.2}$$



(a) Elevation view.



(b) Plan view.



(c) Cross-section.

Figure 9.4.1 An elemental reach of channel for derivation of Saint–Venant equations.

This is negative because inflows are considered negative in the control volume approach (Reynolds transport theorem). The mass outflow from the control volume is

$$\int_{\text{outlet}} \rho \mathbf{V} \, \mathbf{dA} = \rho\left(Q + \frac{\partial Q}{\partial x} dx\right) \tag{9.4.3}$$

where $\partial Q/\partial x$ is the rate of change of channel flow with distance. The volume of the channel element is Adx, where A is the average cross-sectional area, so the rate of change of mass stored within the control volume is

$$\frac{d}{dt}\int_{\text{CV}} \rho \, d\forall = \frac{\partial(\rho A dx)}{\partial t} \tag{9.4.4}$$

where the partial derivative is used because the control volume is defined to be fixed in size (though the water level may vary within it). The net outflow of mass from the control volume is found by substituting equations (9.4.2)–(9.4.4) into (9.4.1):

$$\frac{\partial(\rho A dx)}{dt} - \rho(Q + q dx) + \rho\left(Q + \frac{\partial Q}{\partial x}dx\right) = 0 \tag{9.4.5}$$

Assuming the fluid density ρ is constant, equation (9.4.5) is simplified by dividing through by ρdx and rearranging to produce the *conservation form* of the continuity equation,

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} - q = 0 \tag{9.4.6}$$

which is applicable at a channel cross-section. This equation is valid for a *prismatic* or a *nonprismatic* channel; a prismatic channel is one in which the cross-sectional shape does not vary along the channel and the bed slope is constant.

For some methods of solving the Saint–Venant equations, the *nonconservation form* of the continuity equation is used, in which the average flow velocity V is a dependent variable, instead of Q. This form of the continuity equation can be derived for a unit width of flow within the channel, neglecting lateral inflow, as follows. For a unit width of flow, $A = y \times 1 = y$ and $Q = VA = Vy$. Substituting into equation (9.4.6) yields

$$\frac{\partial (Vy)}{\partial x} + \frac{\partial y}{\partial t} = 0 \tag{9.4.7}$$

or

$$V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} + \frac{\partial y}{\partial t} = 0 \tag{9.4.8}$$

9.4.2 Momentum Equation

Newton's second law is written in the form of Reynolds transport theorem as in equation (3.4.5):

$$\sum \mathbf{F} = \frac{d}{dt}\int_{CV} \mathbf{V}\rho\, d\forall + \sum_{CS} \mathbf{V}\rho\mathbf{V}\cdot\mathbf{dA} \tag{9.4.9}$$

This states that the sum of the forces applied is equal to the rate of change of momentum stored within the control volume plus the net outflow of momentum across the control surface. This equation, in the form $\Sigma F = 0$, was applied to steady uniform flow in an open channel in Chapter 5. Here, unsteady nonuniform flow is considered.

Forces. There are five forces acting on the control volume:

$$\sum F = F_g + F_f + F_e + F_p \tag{9.4.10}$$

where F_g is the *gravity force* along the channel due to the weight of the water in the control volume, F_f is the *friction force* along the bottom and sides of the control volume, F_e is the *contraction/expansion force* produced by abrupt changes in the channel cross-section, and F_p is the *unbalanced pressure force* (see Figure 9.4.1). Each of these four forces is evaluated in the following paragraphs.

Gravity. The volume of fluid in the control volume is Adx and its weight is $\rho gAdx$. For a small angle of channel inclination θ, $S_0 \approx \sin\theta$ and the gravity force is given by

$$F_g = \rho gAdx \sin\theta \approx \rho gAS_0dx \tag{9.4.11}$$

where the channel bottom slope S_0 equals $-\partial z/\partial x$.

Friction. Frictional forces created by the shear stress along the bottom and sides of the control volume are given by $-\tau_0 P dx$, where $\tau_0 = \gamma R S_f = \rho g (A/P) S_f$ is the bed shear stress and P is the wetted perimeter. Hence the friction force is written as

$$F_f = -\rho g A S_f dx \tag{9.4.12}$$

where the friction slope S_f is derived from resistance equations such as Manning's equation.

Contraction/expansion. Abrupt contractions or expansions of the channel cause energy losses through eddy motion. Such losses are similar to minor losses in a pipe system. The magnitude of eddy losses is related to the change in velocity head $V^2/2g = (Q/A)^2/2g$ through the length of channel causing the losses. The drag forces creating these eddy losses are given by

$$F_e = -\rho g A S_e dx \tag{9.4.13}$$

where S_e is the eddy loss slope

$$S_e = \frac{K_e}{2g}\frac{\partial (Q/A)^2}{\partial x} \tag{9.4.14}$$

in which K_e is the nondimensional expansion or contraction coefficient, negative for channel expansion (where $\partial(Q/A)^2/\partial x$ is negative) and positive for channel contractions.

Pressure. Referring to Figure 9.4.1, the unbalanced pressure force is the resultant of the hydrostatic force on the each side of the control volume. Chow et al. (1988) provide a detailed derivation of the pressure force F_p as simply

$$F_p = \rho g A \frac{\partial y}{\partial x} dx \tag{9.4.15}$$

The sum of the forces in equation (9.4.10) can be expressed, after substituting equations (9.4.11), (9.4.12), (9.4.13), and (9.4.15), as

$$\Sigma F = \rho A S_0 dx - \rho g A S_f dx - \rho g A S_e dx - \rho g A \frac{\partial y}{\partial x} dx \tag{9.4.16}$$

Momentum. The two momentum terms on the right-hand side of equation (9.4.9) represent the rate of change of storage of momentum in the control volume, and the net outflow of momentum across the control surface, respectively.

Net momentum outflow. The mass inflow rate to the control volume (equation (9.4.2)) is $-\rho(Q + qdx)$, representing both stream inflow and lateral inflow. The corresponding momentum is computed by multiplying the two mass inflow rates by their respective velocity and a *momentum correction factor* β:

$$\int_{\text{inlet}} \mathbf{V}\rho\mathbf{V}\,\mathbf{dA} = -\rho\left(\beta V Q + \beta v_x q dx\right) \tag{9.4.17}$$

where $-\rho\beta VQ$ is the momentum entering from the upstream end of the channel, and $-\rho\beta v_x q dx$ is the momentum entering the main channel with the lateral inflow, which has a velocity v_x in the x direction. The term β is known as the *momentum coefficient* or *Boussinesq coefficient*; it accounts for the nonuniform distribution of velocity at a channel cross-section in computing the momentum. The value of β is given by

$$\beta = \frac{1}{V^2 A}\int v^2 dA \tag{9.4.18}$$

where v is the velocity through a small element of area dA in the channel cross-section. The value of β ranges from 1.01 for straight prismatic channels to 1.33 for river valleys with floodplains (Chow, 1959; Henderson, 1966).

The momentum leaving the control volume is

$$\int_{\text{outlet}} \mathbf{V}\rho\mathbf{V}\,\mathbf{dA} = \rho\left[\beta VQ + \frac{\partial(\beta VQ)}{\partial x}dx\right] \tag{9.4.19}$$

The net outflow of momentum across the control surface is the sum of equations (9.4.17) and (9.4.19):

$$\int_{\text{CS}} \mathbf{V}\rho\mathbf{V}\mathbf{dA} = -\rho(\beta VQ + \beta v_x q dx) + \rho\left[\beta VQ + \frac{\partial(\beta VQ)}{\partial x}dx\right]$$

$$= -\rho\left[\beta v_x q - \frac{\partial(\beta VQ)}{\partial x}\right]dx \tag{9.4.20}$$

Momentum storage. The time rate of change of momentum stored in the control volume is found by using the fact that the volume of the elemental channel is Adx, so its momentum is $\rho AdxV$, or ρQdx, and then

$$\frac{d}{dt}\int_{\text{CV}} \mathbf{V}\rho d\forall = \rho\frac{\partial Q}{\partial x}dx \tag{9.4.21}$$

After substituting the force terms from equation (9.4.16) and the momentum terms from equations (9.4.20) and (9.4.21) into the momentum equation (9.4.9), it reads

$$\rho gAS_0dx - \rho gAS_fdx - \rho gAS_edx - \rho gA\frac{\partial y}{\partial x}dx = -\rho\left[\beta\, v_x q - \frac{\partial(\beta VQ)}{\partial x}\right]dx + \rho\frac{\partial Q}{\partial t}dx \tag{9.4.22}$$

Dividing through by ρdx, replacing V with Q/A, and rearranging produces the conservation form of the momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial(\beta Q^2/A)}{\partial t} + gA\left(\frac{\partial y}{\partial x} - S_0 + S_f + S_e\right) - \beta q v_x = 0 \tag{9.4.23}$$

The depth y in equation (9.4.23) can be replaced by the water surface elevation h, using

$$h = y + z \tag{9.4.24}$$

where z is the elevation of the channel bottom above a datum such as mean sea level. The derivative of equation (9.4.24) with respect to the longitudinal distance x along the channel is

$$\frac{\partial h}{\partial x} = \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} \tag{9.4.25}$$

but $\partial z/\partial x = -S_0$, so

$$\frac{\partial h}{\partial x} = \frac{\partial y}{\partial x} - S_0 \tag{9.4.26}$$

The momentum equation can now be expressed in terms of h by using equation (9.4.26) in (9.4.23):

$$\frac{\partial Q}{\partial t} + \frac{\partial\left(\beta\ Q^2/A\right)}{\partial x} + gA\left(\frac{\partial h}{\partial x} + S_f + S_e\right) - \beta q v_x = 0 \tag{9.4.27}$$

The Saint–Venant equations, (9.4.6) for continuity and (9.4.27) for momentum, are the governing equations for one-dimensional, unsteady flow in an open channel. The use of the terms S_f and S_e in equation (9.4.27), which represent the rate of energy loss as the flow passes through the channel, illustrates the close relationship between energy and momentum considerations in describing the flow. Strelkoff (1969) showed that the momentum equation for the Saint–Venant equations can also be derived from energy principles, rather than by using Newton's second law as presented here.

The nonconservation form of the momentum equation can be derived in a similar manner to the nonconservation form of the continuity equation. Neglecting eddy losses, wind shear effect, and lateral inflow, the nonconservation form of the momentum equation for a unit width in the flow is

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\left(\frac{\partial y}{\partial x} - S_0 + S_f\right) = 0 \tag{9.4.28}$$

9.5 KINEMATIC WAVE MODEL FOR CHANNELS

In Section 8.9, a kinematic wave overland flow runoff model was presented. This is an implicit nonlinear kinematic model that is used in the KINEROS model. This section presents a general discussion of the kinematic wave followed by brief description of the very simplest linear models, such as those found in the U.S. Army Corps of Engineers HEC-1, and the more complicated models such as the KINEROS model (Woolhiser et al., 1990).

Kinematic waves govern flow when inertial and pressure forces are not important. Dynamic waves govern flow when these forces are important, as in the movement of a large flood wave in a wide river. In a kinematic wave, the gravity and friction forces are balanced, so the flow does not accelerate appreciably.

For a kinematic wave, the energy grade line is parallel to the channel bottom and the flow is steady and uniform ($S_0 = S_f$) within the differential length, while for a dynamic wave the energy grade line and water surface elevation are not parallel to the bed, even within a differential element.

9.5.1 Kinematic Wave Equations

A *wave* is a variation in a flow, such as a change in flow rate or water surface elevation, and the *wave celerity* is the velocity with which this variation travels along the channel. The celerity depends on the type of wave being considered and may be quite different from the water velocity. For a kinematic wave the acceleration and pressure terms in the momentum equation are negligible, so the wave motion is described principally by the equation of continuity. The name kinematic is thus applicable, as *kinematics* refers to the study of motion exclusive of the influence of mass and force; in *dynamics* these quantities are included.

The kinematic wave model is defined by the following equations.

Continuity:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q(x, t) \tag{9.5.1}$$

Momentum:

$$S_0 = S_f \tag{9.5.2}$$

where $q(x, t)$ is the net lateral inflow per unit length of channel.

The momentum equation can also be expressed in the form

$$A = \alpha Q^{\beta} \tag{9.5.3}$$

For example, Manning's equation written with $S_0 = S_f$ and $R = A/P$ is

$$Q = \frac{1.49 S_0^{1/2}}{n P^{2/3}} A^{5/3} \tag{9.5.4}$$

which can be solved for A as

$$A = \left(\frac{n P^{2/3}}{1.49 \sqrt{S_0}} \right)^{3/5} Q^{3/5} \tag{9.5.5}$$

so $\alpha = \left[nP^{2/3} / \left(1.49 \sqrt{S_0} \right) \right]^{0.6}$ and $\beta = 0.6$ in this case.

Equation (9.5.1) contains two dependent variables, A and Q, but A can be eliminated by differentiating equation (9.5.3):

$$\frac{\partial A}{\partial t} = \alpha \beta Q^{\beta - 1} \left(\frac{\partial Q}{\partial t} \right) \tag{9.5.6}$$

and substituting for $\partial A / \partial t$ in equation (9.5.1) to give

$$\frac{\partial Q}{\partial x} + \alpha \beta Q^{\beta - 1} \left(\frac{\partial Q}{\partial t} \right) = q \tag{9.5.7}$$

Alternatively, the momentum equation could be expressed as

$$Q = aA^B \tag{9.5.8}$$

where a and B are defined using Manning's equation. Using

$$\frac{\partial Q}{\partial x} = \frac{dQ}{dA} \frac{\partial A}{\partial x} \tag{9.5.9}$$

the governing equation is

$$\frac{\partial A}{\partial t} + \frac{dQ}{dA} \frac{\partial A}{\partial x} = q \tag{9.5.10}$$

where dQ/dA is determined by differentiating equation (9.5.8):

$$\frac{dQ}{dA} = aBA^{B-1} \tag{9.5.11}$$

and substituting in equation (9.5.10):

$$\frac{\partial A}{\partial t} = aBA^{B-1} \frac{\partial A}{\partial t} = q \tag{9.5.12}$$

The kinematic wave equation (9.5.7) has Q as the dependent variable and the kinematic wave equation (9.5.12) has A as the dependent variable. First consider equation (9.5.7), by taking the logarithm of (9.5.3):

$$\ln A = \ln \alpha + \beta \ln Q \tag{9.5.13}$$

and differentiating

$$\frac{dQ}{Q} = \frac{1}{\beta} \left(\frac{dA}{A} \right) \tag{9.5.14}$$

This defines the relationship between relative errors dA/A and dQ/Q. For Manning's equation $\beta < 1$, so that the discharge estimation error would be magnified by the ratio $1/\beta$ if A were the dependent variable instead of Q.

Next consider equation (9.5.12); by taking the logarithm of (9.5.8):

$$\ln Q = \ln a + B \ln A \tag{9.5.15}$$

$$\frac{dA}{A} = \frac{1}{B}\left(\frac{dQ}{Q}\right)$$

or

$$\frac{dQ}{Q} = B\left(\frac{dA}{A}\right) \tag{9.5.16}$$

In this case $B > 1$, so that the discharge estimation error would be decreased by B if A were the dependent variable instead of Q. In summary, if we use equation (9.5.3) as the form of the momentum equation, then Q is the dependent variable with equation (9.5.7) being the governing equation; if we use equation (9.5.8) as the form of the momentum equation, then A is the dependent variable with equation (9.5.12) being the governing equation.

9.5.2 U.S. Army Corps of Engineers HEC-1 Kinematic Wave Model for Overland Flow and Channel Routing

The HEC-1 computer program actually has two forms of the kinematic wave. The first is based upon equation (9.5.12) where an explicit finite difference form is used (refer to Figures 9.5.1 and 8.9.2):

$$\frac{\partial A}{\partial t} = \frac{A_{i+1}^{j+1} - A_{i+1}^{j}}{\Delta t} \tag{9.5.17}$$

$$\frac{\partial A}{\partial x} = \frac{A_{i+1}^{j} - A_{i}^{j}}{\Delta x} \tag{9.5.18}$$

and

$$A = \frac{A_{i+1}^{j} + A_{i}^{j}}{2} \tag{9.5.19}$$

$$q = \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} \tag{9.5.20}$$

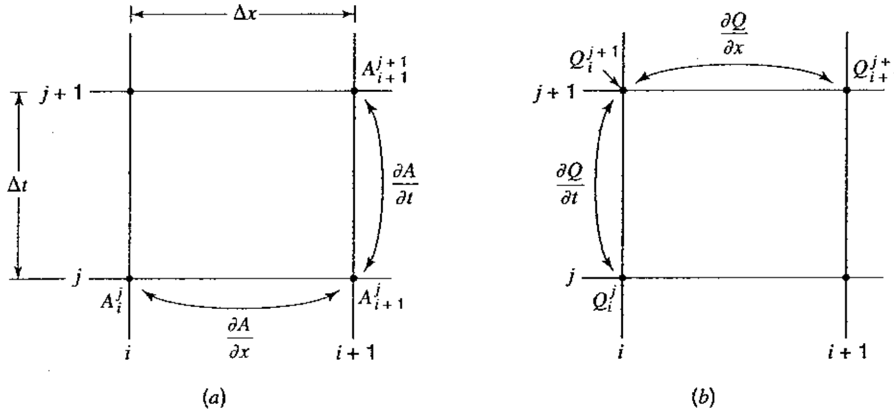


Figure 9.5.1 Finite difference forms. (*a*) HEC-1 "standard form;" (*b*) HEC-1 "conservation form."

Substituting these finite-difference approximations into equation (9.5.12) gives

$$\frac{1}{\Delta t}\left(A_{i+1}^{j+1}-A_{i+1}^{j}\right)+aB\left[\frac{A_{i+1}^{j}+A_{i}^{j}}{2}\right]^{B-1}\left[\frac{A_{i+1}^{j}-A_{i}^{j}}{\Delta x}\right]=\frac{q_{i+1}^{j+1}+q_{i+1}^{j}}{2} \tag{9.5.21}$$

The only unknown in equation (9.5.21) is A_{i+1}^{j+1}, so

$$A_{i+1}^{j+1}=A_{i+1}^{j}-aB\left(\frac{\Delta t}{\Delta x}\right)\left[\frac{A_{i+1}^{j}+A_{i}^{j}}{2}\right]^{B-1}\left(A_{i+1}^{j}-A_{i,j}\right)+\left(q_{i+1}^{j+1}+q_{i+1}^{j}\right)\frac{\Delta t}{2} \tag{9.5.22}$$

After computing A_{i+1}^{j+1} at each grid along a time line going from upstream to downstream (see Figure 8.9.2), compute the flow using equation (9.5.8):

$$Q_{i+1}^{j+1}=a\left(A_{i+1}^{j+1}\right)^{B} \tag{9.5.23}$$

The HEC-1 model uses the above kinematic wave model as long as a stability factor $R < 1$ (Alley and Smith, 1987), defined by

$$R=\frac{a}{q\Delta x}\left[\left(q\Delta t+A_{i}^{j}\right)^{B}-\left(A_{i}^{j}\right)^{B}\right]\ \text{for } q>0 \tag{9.5.24a}$$

$$R=aB\left(A_{i}^{j}\right)^{B-1}\frac{\Delta t}{\Delta x}\ \text{ for } q=0 \tag{9.5.24b}$$

Otherwise HEC-1 uses the form of equation (9.5.1), where (see Figure 9.5.1)

$$\frac{\partial Q}{\partial x}=\frac{Q_{i+1}^{j+1}-Q_{i}^{j+1}}{\Delta x} \tag{9.5.25}$$

$$\frac{\partial A}{\partial t}=\frac{A_{i}^{j+1}-A_{i}^{j}}{\Delta t} \tag{9.5.26}$$

so

$$\frac{Q_{i+1}^{j+1}-Q_{i}^{j+1}}{\Delta x}+\frac{A_{i}^{j+1}-A_{i}^{j}}{\Delta t}=q \tag{9.5.27}$$

Solving for the only unknown Q_{i+1}^{j+1} yields

$$Q_{i+1}^{j+1}=Q_{i}^{j+1}+q\Delta x-\frac{\Delta x}{\Delta t}\left(A_{i}^{j+1}-A_{i}^{j}\right) \tag{9.5.28}$$

Then solve for A_{i+1}^{j+1} using equation (9.5.23):

$$A_{i+1}^{j+1}=\left(\frac{1}{a}Q_{i+1}^{j+1}\right)^{1/B} \tag{9.5.29}$$

The *initial condition* (values of A and Q at time 0 along the grid, referring to Figure 8.9.2) are computed assuming uniform flow or nonuniform flow for an initial discharge. The *upstream boundary* is the inflow hydrograph from which Q is obtained.

The kinematic wave schemes used in the HEC-1 model are very simplified. Chow et al. (1988) presented both linear and nonlinear kinematic wave schemes based upon the equation (9.5.7) formulation. An example of a more desirable kinematic wave formulation is that by Woolhiser et al. (1990) presented in the next subsection.

9.5.3 KINEROS Channel Flow Routing Model

The KINEROS channel routing model uses the equation (9.5.10) form of the kinematic wave equation (Woolhiser et al., 1990):

$$\frac{\partial A}{\partial t}+\frac{dQ}{dA}\frac{\partial A}{\partial x}=q(x,\,t) \tag{9.5.10}$$

where $q(x, t)$ is the net lateral inflow per unit length of channel. The derivatives are approximated using an implicit scheme in which the spatial and temporal derivatives are, respectively,

$$\frac{\partial A}{\partial x}=\theta\frac{A_{i+1}^{j+1}-A_i^{j+1}}{\Delta x}+(1-\theta)\frac{A_{i+1}^{j}-A_i^{j}}{\Delta x} \tag{9.5.30}$$

$$\frac{dQ}{dA}\frac{\partial A}{\partial x}=\theta\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j+1}-A_i^{j+1}}{\Delta x}\right)+(1-\theta)\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j}-A_i^{j}}{\Delta x}\right) \tag{9.5.31}$$

and

$$\frac{\partial A}{\partial t}=\frac{1}{2}\left[\frac{A_i^{j+1}-A_i^{j}}{\Delta t}+\frac{A_{i+1}^{j+1}-A_{i+1}^{j}}{\Delta t}\right] \tag{9.5.32}$$

or

$$\frac{\partial A}{\partial t}=\frac{A_i^{j+1}+A_{i+1}^{j+1}-A_i^{j}-A_{i+1}^{j}}{2\Delta t} \tag{9.5.33}$$

Substituting equations (9.5.31) and (9.5.33) into (9.5.10), we have

$$\frac{A_{i+1}^{j+1}-A_{i+1}^{j}+A_i^{j+1}-A_i^{j}}{2\Delta t}+\left\{\theta\left[\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j+1}-A_i^{j+1}}{\Delta x}\right)\right]+(1-\theta)\left[\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j}-A_i^{j}}{\Delta x}\right)\right]\right\}$$

$$=\frac{1}{2}\left(q_{i+1}^{j+1}+q_i^{j+1}+q_{i+1}^{j}+q_i^{j}\right) \tag{9.5.34}$$

The only unknown in this equation is A_{i+1}^{j+1}, which must be solved for numerically by use of an iterative scheme such as the Newton–Rhapson method (see Appendix A).

Woolhiser et al. (1990) use the following relationship between channel discharge and cross-sectional area, which embodies the kinematic wave assumption:

$$Q=\alpha R^{m-1}A \tag{9.5.35}$$

where R is the hydraulic radius and $\alpha = 1.49S^{1/2}/n$ and $m = 5/3$ for Manning's equation.

9.5.4 Kinematic Wave Celerity

Kinematic waves result from changes in Q. An increment in flow dQ can be written as

$$dQ=\frac{\partial Q}{\partial x}dx+\frac{\partial Q}{\partial t}dt \tag{9.5.36}$$

Dividing through by dx and rearranging produces:

$$\frac{\partial Q}{\partial x}+\frac{dt}{dx}\frac{\partial Q}{\partial t}=\frac{dQ}{dx} \tag{9.5.37}$$

Equations (9.5.7) and (9.5.37) are identical if

$$\frac{dQ}{dt} = q \tag{9.5.38}$$

and

$$\frac{dx}{dt} = \frac{1}{\alpha\beta Q^{\beta-1}} \tag{9.5.39}$$

Differentiating equation (9.5.3) and rearranging gives

$$\frac{dQ}{dA} = \frac{1}{\alpha\beta Q^{\beta-1}} \tag{9.5.40}$$

and by comparing equations (9.5.38) and (9.5.40), it can be seen that

$$\frac{dx}{dt} = \frac{dQ}{dA} \tag{9.5.41}$$

or

$$c_k = \frac{dx}{dt} = \frac{dQ}{dA} \tag{9.5.42}$$

where c_k is the kinematic wave celerity. This implies that an observer moving at a velocity $dx/dt = c_k$ with the flow would see the flow rate increasing at a rate of $dQ/dx = q$. If $q = 0$ the observer would see a constant discharge. Equations (9.5.38) and (9.5.42) are the *characteristic equations* for a kinematic wave, two ordinary differential equations that are mathematically equivalent to the governing continuity and momentum equations.

The kinematic wave celerity can also be expressed in terms of the depth y as

$$c_k = \frac{1}{B}\frac{dQ}{dy} \tag{9.5.43}$$

where $dA = Bdy$.

Both kinematic and dynamic wave motion are present in natural flood waves. In many cases the channel slope dominates in the momentum equation; therefore, most of a flood wave moves as a kinematic wave. Lighthill and Whitham (1955) proved that the velocity of the main part of a natural flood wave approximates that of a kinematic wave. If the other momentum terms ($\partial V/\partial t$, $V(\partial V/\partial x)$ and $(1/g)\partial y/\partial x$) are not negligible, then a dynamic wave front exists that can propagate both upstream and downstream from the main body of the flood wave.

9.6 MUSKINGUM-CUNGE MODEL

Cunge (1969) proposed a variation of the kinematic wave method based upon the Muskingum method (see Chapter 8). With the grid shown in Figure 9.6.1, the unknown discharge Q_{i+1}^{j+1} can be expressed using the Muskingum equation ($Q_{j+1} = C_1 I_{j+1} + C_2 I_j + C_3 Q_j$):

$$Q_{i+1}^{j+1} = C_1 Q_i^{j+1} + C_2 Q_i^j + C_3 Q_{i+1}^j \tag{9.6.1}$$

where $Q_{i+1}^{j+1} = Q_{j+1}$; $Q_i^{j+1} = I_{j+1}$; $Q_i^j = I_j$; and $Q_{i+1}^j = Q_j$. The Muskingum coefficients are

$$C_1 = \frac{\Delta t - 2KX}{2K(1-X) + \Delta t} \tag{9.6.2}$$

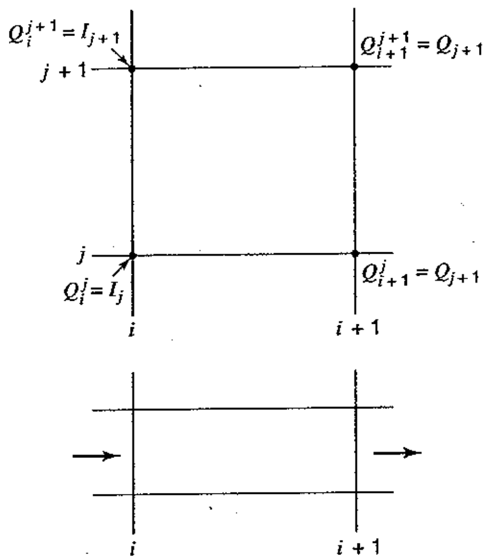


Figure 9.6.1 Finite-difference grid for Muskingum–Cunge method.

$$C_2 = \frac{\Delta t + 2KX}{2K(1-X)+\Delta t} \tag{9.6.3}$$

$$C_3 = \frac{2K(1-X)-\Delta t}{2K(1-X)+\Delta t} \tag{9.6.4}$$

Cunge (1969) showed that when K and Δt are considered constant, equation (9.6.10) is an approximate solution of the kinematic wave. He further demonstrated that (9.6.1) can be considered an approximation of a modified diffusion equation if

$$K = \frac{\Delta x}{c_k} = \frac{\Delta x}{dQ/dA} \tag{9.6.5}$$

and

$$X = \frac{1}{2}\left(1 - \frac{Q}{Bc_k S_0 \Delta x}\right) \tag{9.6.6}$$

where c_k is the celerity corresponding to Q and B, and B is the width of the water surface. The value of $\Delta x/(dQ/dA)$ in equation (9.6.5) represents the time propagation of a given discharge along a channel reach of length Δx. Numerical stability requires $0 \le x \le 1/2$. The solution procedure is basically the same as the kinematic wave.

9.7 IMPLICIT DYNAMIC WAVE MODEL

The conservation form of the Saint–Venant equations is used because this form provides the versatility required to simulate a wide range of flows from gradual long-duration flood waves in rivers to abrupt waves similar to those caused by a dam failure. The equations are developed from equations (9.4.6) and (9.4.25) as follows.

Weighted four-point finite-difference approximations given by equations (9.7.1)–(9.7.3) are used for dynamic routing with the Saint–Venant equations. The spatial derivatives $\partial Q/\partial x$ and $\partial h/\partial x$ are estimated between adjacent time lines:

$$\frac{\partial Q}{\partial x} = \theta \frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x_i} + (1-\theta)\frac{Q_{i+1}^{j} - Q_i^{j}}{\Delta x_i} \tag{9.7.1}$$

$$\frac{\partial h}{\partial x} = \theta \frac{h_{i+1}^{j+1} - h_i^{j+1}}{\Delta x_i} + (1-\theta)\frac{h_{i+1}^{j} - h_i^{j}}{\Delta x_i} \tag{9.7.2}$$

and the time derivatives are:

$$\frac{\partial (A+A_0)}{\partial t} = \frac{(A+A_0)_i^{j+1} + (A+A_0)_{i+1}^{j+1} - (A+A_0)_i^{j} - (A+A_0)_{i+1}^{j}}{2\Delta t_j} \tag{9.7.3}$$

$$\frac{\partial Q}{\partial t} = \frac{Q_i^{j+1} + Q_{i+1}^{j+1} - Q_i^{j} - Q_{i+1}^{j}}{2\Delta t_j} \tag{9.7.4}$$

The nonderivative terms, such as q and A, are estimated between adjacent time lines, using:

$$q = \theta \frac{q_i^{j+1} + q_{i+1}^{j+1}}{2} + (1-\theta)\frac{q_i^{j} + q_{i+1}^{j}}{2} = \theta \bar{q}_i^{j+1} + (1-\theta)\bar{q}_i^{j} \tag{9.7.5}$$

$$A = \theta\left[\frac{A_i^{j+1} + A_{i+1}^{j+1}}{2}\right] + (1-\theta)\left[\frac{A_i^{j} + A_{i+1}^{j}}{2}\right] = \theta \bar{A}_i^{j+1} + (1-\theta)\bar{A}_i^{j} \tag{9.7.6}$$

where $\bar{q}_i$ and $\bar{A}_i$ indicate the lateral flow and cross-sectional area averaged over the reach Δx_i.

The finite-difference form of the continuity equation is produced by substituting equations. (9.7.1), (9.7.3), and (9.7.5) into (9.4.6):

$$\theta\left(\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x_i} - \bar{q}_i^{j+1}\right) + (1-\theta)\left(\frac{Q_{i+1}^{j} - Q_i^{j}}{\Delta x_i} - \bar{q}_i^{j}\right)$$
$$+ \frac{(A+A_0)_i^{j+1} + (A+A_0)_{i+1}^{j+1} - (A+A_0)_i^{j} - (A+A_0)_{i+1}^{j}}{2\Delta t_j} = 0 \tag{9.7.7}$$

Similarly, the momentum equation (9.4.27) is written in finite-difference form as:

$$\frac{Q_i^{j+1} + Q_{i+1}^{j+1} - Q_i^{j} - Q_{i+1}^{j}}{2\Delta t_j}$$
$$+\theta\left[\frac{(\beta Q^2/A)_i^{j+1} - (\beta Q^2/A)_{i+1}^{j+1}}{\Delta x_i} + g\bar{A}_i^{j+1}\left(\frac{h_{i+1}^{j+1} - h_i^{j+1}}{\Delta x_i} + (\bar{S}_f)_i^{j+1} + (\bar{S}_e)_i^{j+1}\right) - (\overline{\beta q v_x})_i^{j+1}\right.$$
$$\left.+(1-\theta)\left[\frac{(\beta Q^2/A)_{i+1}^{j} - (\beta Q^2/A)_{i+1}^{j}}{\Delta x_i} + g\bar{A}_i^{j}\left(\frac{h_{i+1}^{j} - h_i^{j}}{\Delta x_i} + (\bar{S}_f)_i^{j} + (\bar{S}_e)_i^{j}\right) - (\overline{\beta q v_x})_i^{j}\right]\right] = 0 \tag{9.7.8}$$

The four-point finite-difference form of the continuity equation can be further modified by multiplying equation (9.7.7) by Δx_i to obtain

$$\theta(Q_{i+1}^{j+1} - Q_i^{j+1} - \bar{q}_i^{j+1}\Delta x_i) + (1-\theta)(Q_{i+1}^{j} - Q_i^{j} - \bar{q}_i^{j}\Delta x_i)$$
$$+\frac{\Delta x_i}{2\Delta t_i}\left[(A+A_0)_i^{j+1} + (A+A_0)_{i+1}^{j+1} - (A+A_0)_i^{j} - (A+A_0)_{i+1}^{j}\right] = 0 \tag{9.7.9}$$