Scaling and Similarity in Rough Channel Flows

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We show that Manning's empirical formula for the mean velocity of turbulent flows in channels represents the power-law asymptotic behavior of a flow of incomplete similarity in the relative roughness. We then derive the formula based on the phenomenological theory of turbulence. Our derivation yields the correct similarity exponent; it justifies Manning's use of a single parameter, the hydraulic radius, to characterize the geometry of the cross section; and it affords insight into the mechanism of momentum transfer.

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Manning's empirical formula for the mean velocity of gravity-driven, uniform, fully developed turbulent flows in rough open channels is among the better known expressions used by hydrologists, geomorphologists, and hydraulic engineers. The formula is customarily used to determine the capacity of natural streams and flood plains, and to design artificial channels [1,2]. It has also been used to quantify the vast flows which appear to have occurred on Mars in pre-Amazonian times [3]. Because it embodies a large corpus of experimental results [1], and it is known to work very well, Manning's formula affords a singular opportunity for gaining insight into a problem of considerable theoretical interest and numerous applications. Yet, there exists no theory of Manning's formula, and the following assertion, made in a classical text on geomorphology [4], remains valid after thirty-seven years: "It is truly surprising that engineering practice has depended to such an extent on a formula as empirical as this one, derived nearly a century ago."

Manning's formula is usually written in the dimensionally inconsistent form

$$V = \frac{1}{n} s^{1/2} R^{2/3}, \tag{1}$$

where V is the mean velocity of the flow, s is the slope of the channel, R is the *hydraulic radius* of the cross section of the channel, and n is the *roughness coefficient*. The hydraulic radius is defined as the area of the cross section of the channel divided by the length of the wetted perimeter; for example, for a rectangular channel of width b and depth h, $R = bh/(b + 2h)$, and $\lim_{b\to\infty} R = h$. (The use of the single parameter R to characterize the geometry of the cross section has been amply verified experimentally, at least for the case of rectangular channels.) Appropriate values of n have been measured for different types of channel walls, and tabulated [2,5].

Before deriving Manning's formula, Eq. (1), we ascertain to what extent it can be predicated on dimensional analysis and suitable assumptions of similarity. Based on (1), we start by including V, R, and gs in our set of variables, where gs is the active component of the gravitational acceleration. To characterize the roughness of the channel walls we follow several authors (e.g., [2]) in using a variable r, the *absolute roughness,* which has units of length. The dimensional equations $[gs] = [V^2/R]$ and $[r] = [R]$ show that the dimensions of two of the variables (gs and r) can be expressed as products of powers of the dimensions of the other variables; it follows from Buckingham's Π theorem [6] that we can reduce the functional relationship among V, R, gs, and r to an equivalent functional relationship between two dimensionless variables. A sensible choice of dimensionless variables is $F \equiv V/\sqrt{gR}$ (the Froude number) and r/R (the relative roughness). We express the functional relationship between F and the relative roughness in the form $F = \mathcal{F}[r/R]$, or, equivalently,

$$V = \mathcal{F}\left[\frac{r}{R}\right]\sqrt{Rgs}, \tag{2}$$

where $\mathcal{F}$ is a dimensionless function of r/R. To make further progress, we note that in rivers and artificial channels $r/R \ll 1$, and seek to formulate an asymptotic similarity law for $r/R \to 0$. There are two possible similarities: complete and incomplete [6]. In the case of complete similarity in r/R, $\mathcal{F}[r/R]$ tends to a constant as $r/R \to 0$. This would make V independent of the roughness in rivers and artificial channels (where $r/R \ll 1$), which is contrary to experimental observation. In the case of incomplete similarity in r/R, (2) admits the following power-law asymptotics [6],

$$V = K\left(\frac{r}{R}\right)^{\alpha}\sqrt{Rgs} + o\left[\left(\frac{r}{R}\right)^{\alpha}\right], \tag{3}$$

where K is a dimensionless constant, and α is a similarity exponent, which cannot be determined by dimensional analysis. A comparison of (3) with (1) shows that the leading term of (3) is compatible with Manning's formula, and that $\alpha = -1/6$. The value of α is the most important empirical result implicit in Manning's formula. A comparison of (3) with (1) also shows that $n = K^{-1} r^{1/6} g^{-1/2}$. Two pieces of this expression for n have been proposed previously. The scaling $n \sim g^{-1/2}$ was suggested by Chow [2], and later justified dimensionally by Yen [7]; it was first used by Carr [3] to adapt the tabulated values of n to the gravitational field of Mars. The scaling $n \sim r^{1/6}$ was proposed by Strickler [2] based on the analysis of extensive experimental data.

In deriving Manning's formula we expect to verify (i) that an incomplete similarity in r/R prevails for $r/R \ll 1$; (ii) that the similarity exponent is $\alpha = -1/6$; and (iii) that for rectangular channels the hydraulic radius suffices to characterize the geometry of the cross section.

We start by considering a rectangular channel of slope s. Then, the streamwise component of the gravitational force per unit length of channel is $F_g = \rho bhgs$, where ρ is the density of the fluid. Let us call S a wetted surface tangent to the peaks of the roughness elements, Fig. 1. (For the time being, we need only consider roughness elements of uniform size r.) Under conditions of fully developed turbulence, the streamwise component of the force on S per unit length of channel is $F_\tau = (b + 2h)\tau$. In this expression, $b + 2h$ is the wetted perimeter and $\tau = \rho|\overline{v_n v_t}|$ is a Reynolds shear stress, where v_n and v_t are the fluctuating velocities normal and tangent to S, respectively, and an overbar denotes time average. We study v_n first, and start by making a crucial observation: when the relative roughness is small $(r/R \ll 1)$, turbulent eddies of sizes larger than, say, $2r$, can provide only a negligible velocity normal to S, Fig. 1. On the other hand, turbulent eddies smaller than r fit in the space between successive roughness elements, and they can provide a velocity normal to S. However, when these eddies are smaller than, say, $r/2$, their characteristic velocities are negligible compared with the characteristic velocity of the eddies of size r. Thus, v_n is dominated by u_r, which is the characteristic velocity associated with the eddies of size r (a suitable mathematical expression for u_r is given in [8]). In other words, $v_n \sim u_r$, where the symbol "$\sim$" means "scales with." We now turn to v_t. Eddies of all sizes can provide a velocity tangent to S. It follows that v_t is dominated by V, which is the characteristic velocity associated with the largest eddies, and $v_t \sim V$. We surmise that $|\overline{v_n v_t}| \sim u_r V$, which together with the equation of balance of momentum transfer, $F_g = F_\tau$, leads to

$$u_r V \sim \left(\frac{bh}{b + 2h}\right) gs = Rgs \,. \qquad (4)$$

We now seek to relate u_r and V. To that end we use Kolmogórov's scaling. This scaling can be easily derived for isotropic turbulence. It has been proved, however, that the scaling applies as well to turbulence which is not only anisotropic, but also inhomogeneous [9] (the turbulence is inhomogeneous in the vicinity of the wall). If the eddies of size r are within the inertial range (i.e., if $r \gg \eta$, where η is the Kolmogórov length), then $u_r^3/r \sim \varepsilon$, where ε is the rate of dissipation of turbulent energy per unit mass. According to Kolmogórov's theory of turbulence, ε equals the rate of production of turbulent energy per unit mass, and is independent of the viscosity [8,10]. It follows that a scaling expression for ε can be obtained in terms of V, b, and h. The largest eddies possess an energy per unit mass $\sim V^2$; of these, the ones with horizontal vorticity vector are characterized by a turnover time h/V, whereas the ones with vertical vorticity vector are characterized by a turnover time $(b/2)/V$, Fig. 2. We conclude that

$$\frac{u_r^3}{r} \sim \varepsilon \sim \frac{V^2}{h/V} + \frac{V^2}{b/2V} = \left(\frac{b + 2h}{bh}\right) V^3 = \frac{V^3}{R} \,, \qquad (5)$$

whereupon

$$u_r \sim \left(\frac{r}{R}\right)^{1/3} V \,. \qquad (6)$$

This equation indicates that u_r is self-similar in r with exponent $1/3$, a well-known result of Kolmogórov's theory [8]. More surprisingly, r appears normalized by the hydraulic radius R. Substituting (6) into (4) yields

$$V \sim \left(\frac{r}{R}\right)^{-1/6} \sqrt{Rgs} \,, \qquad (7)$$

which is the leading term of (3) with $\alpha = -1/6$, as expected. This concludes our derivation.

We have derived Manning's formula for the case of channel walls with roughness elements of uniform size r. We now generalize our derivation to the case of channel walls with roughness elements in a range of sizes. Consider

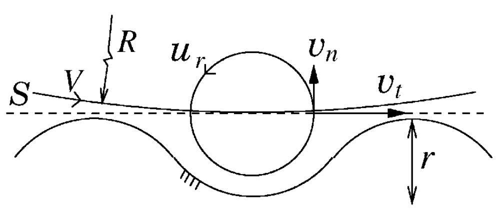

FIG. 1. Immediate vicinity of a channel wall with roughness elements of characteristic size r. The dashed line is the trace of a wetted surface S tangent to the peaks of the roughness elements.

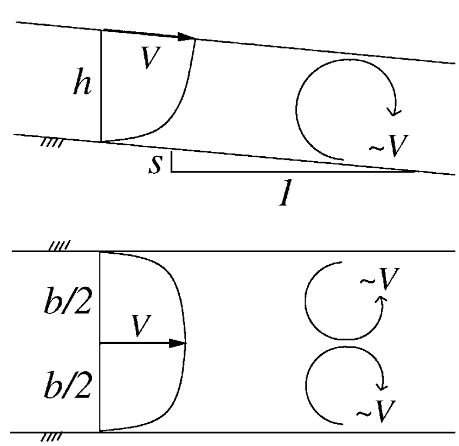

FIG. 2. Largest-length-scale eddies in a rectangular channel of width b and depth h. The velocity of these eddies scales with the mean velocity of the flow V.

a channel wall $\mathcal{W}_1$ characterized by a probability distribution $p(\sigma)$, $\int_0^\infty p(\sigma)\,d\sigma = 1$, where $p(\sigma)\,d\sigma$ measures the probability of finding a roughness element of a size between σ and $\sigma + d\sigma$. Assume that for the channel wall $\mathcal{W}_1$ the average roughness element is of size 1, i.e., that $\int_0^\infty \sigma p(\sigma)\,d\sigma = 1$. Then, we can use $\mathcal{W}_1$ to generate a family of geometrically similar channel-wall surfaces $\{\mathcal{W}_r\}$. For a generic member $\mathcal{W}_r$ of this family of channel-wall surfaces the average roughness element is of size r, i.e., $\int_0^\infty \sigma p(\sigma/r)\,d(\sigma/r) = r$. (The concept of geometrically similar channel-wall surfaces dates back to the early Twentieth Century; see, e.g., [11]). We now rederive Manning's formula for a generic member $\mathcal{W}_r$ of the family of geometrically similar channel-wall surfaces $\{\mathcal{W}_r\}$. The average Reynolds stress on the wetted surface S of Fig. 1 is $\tau = \rho V \int_0^\infty u_\sigma p(\sigma/r)\,d(\sigma/r)$, and we can rewrite (4) in the form

$$V \int_0^\infty u_\sigma p(\sigma/r)\,d(\sigma/r) \sim Rgs\,. \qquad (8)$$

On the other hand,

$$u_\sigma \sim \left(\frac{\sigma}{R}\right)^{1/3} V = \left(\frac{\sigma}{r}\right)^{1/3} \left(\frac{r}{R}\right)^{1/3} V\,. \qquad (9)$$

Substituting (9) into (8) leads to the leading term of (3) with

$$K = K_0 \left(\int_0^\infty \xi^{1/3} p(\xi)\,d\xi \right)^{-1/2}, \qquad (10)$$

where K_0 is a constant.

We obtained (7) based on three assumptions. The first one is that $r/R \ll 1$. In keeping with this assumption, (7) corresponds to the leading term in the power-law asymptotics of Eq. (3). The second assumption is that the turbulent eddies in the vicinity of the walls are governed by Kolmogórov's scaling (6). This is justified because Kolmogórov's scaling has been shown to apply to inhomogeneous turbulence. The third assumption is that the spaces between roughness elements are occupied by eddies of size r, in the form shown in Fig. 1. We now discuss this third assumption.

It is apparent that an eddy of size r *could* be found between any two successive roughness elements. In deriving (7) we have assumed, however, that one such eddy *does* occupy the space between each pair of consecutive roughness elements. Our assumption could be justified by recalling that in Kolmogórov's theory eddies of any given size within the inertial range are space filling (this is required for ε to be scale invariant within the inertial range [8]). It is perhaps more illuminating to think of the assumed set of eddies of size r as akin to the arrays of parallel vortices that have long been documented in the vicinity of smooth channel walls, and which constitute the most common form of *coherent structures*. (Note, however, that the eddy of size r in Fig. 1 need not have a vorticity vector oriented streamwise.) We know from theoretical work on the etiology of coherent structures that numerous instabilities are possible leading to arrays of vortices of specific wavelengths [12,13]. Interestingly, it has been conjectured that the presence of periodic forms of wall roughness (such as, for instance, riblets) may excite instabilities of similar wavelength [12]. This conjecture affords a compelling explanation for the incomplete similarity in the relative roughness, r/R, displayed by Eq. (3). In fact, this similarity is quite puzzling: given that turbulence involves a wide spectrum of wavelengths, spanning many orders of magnitude, why would r, which is just one wavelength somewhere within that spectrum, appear so conspicuously in (3)? The puzzle is explained if the wall roughness induces arrays of eddies of size r in the immediate vicinity of the wall and if, as suggested by our derivation, these eddies effect most of the momentum transfer. Thus, if r diminishes, the capacity for momentum transfer also diminishes; as a result, the fluid friction diminishes, and the mean velocity increases, as indicated by Eq. (7). Given that the size of the eddies is bounded below by the Kolmogórov length η, it is interesting to investigate what happens when r approaches η. To that end, we start by recalling that $\eta = \nu^{3/4}\varepsilon^{-1/4}$, where ν is the kinematic viscosity. From (5) we have $\varepsilon \sim V^3/R$, and therefore $\eta/R \sim (\nu/VR)^{3/4} = \mathrm{Re}^{-3/4}$, where $\mathrm{Re} \equiv VR/\nu$ is the Reynolds number. Therefore, as the roughness approaches the Kolmogórov length (i.e., as the channel walls become hydraulically smooth), we expect (7) to become

$$V \sim \mathrm{Re}^{1/8}\sqrt{Rgs}\,. \qquad (11)$$

The appearance of the Reynolds number in (11) indicates that in the limit $r \to \eta$ the momentum transfer is viscous. It is convenient to write (11) in terms of the *resistance coefficient,* $f \equiv Rgs/V^2$; the result is $f \sim \mathrm{Re}^{-1/4}$, which we recognize as Blasius's classical empirical scaling for hydraulically smooth channels [2]. This result unveils the existence of a relationship among the three well known, and apparently unrelated, scalings due to Blassius, $f \sim \mathrm{Re}^{-1/4}$, Kolmogórov, $\eta = \nu^{3/4}\varepsilon^{-1/4}$, and Manning, $V \sim r^{-1/6}$.

We have provided a derivation of Manning's empirical formula. Besides the final result, we have reached a number of interesting conclusions. For example, in rectangular channels the Reynolds stress in the immediate vicinity of the walls depends on (i) the mean velocity of the flow, (ii) the local wall roughness, and (iii) the depth and width of the cross section through the hydraulic radius only. This conclusion suggests ways of formulating generalized Manning formulas for channels in which different portions of the channel walls are characterized by different families of geometrically similar channel-wall surfaces, of considerable interest in applications. It also has momentous geomorphological implications, since it allows for the determination of absolutely stable aspect ratios, b/h, in natural channels. We shall study these and other related issues in a separate paper.

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